Lattice refinement in loop quantum cosmology

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Abstract. Lattice refinement in LQC, its meaning and its necessity are discussed. The rôle of lattice refinement for the realisation of a successful inflationary model is explicitly shown. A simple and effective numerical technique to solve the constraint equation for any choice of lattice refinement model is briefly illustrated. Phenomenological and consistency requirements leading to a particular choice of lattice refinement model are presented, while it is subsequently proved that only this choice of lattice refinement leads to a unique factor ordering in the Wheeler-De Witt equation, which is the continuum limit of LQC.

1. Introduction

Loop Quantum Gravity (LQG) [1, 2] is one of the most promising candidate theories for describing the quantum degrees of freedom of the gravitational field. Quantum gravity, combining consistently quantum mechanics and general relativity, is essential when curvature becomes large, as for example in the early stages of the evolution of our universe. LQG is a non-perturbative and background independent[1] canonical quantisation of General Relativity (GR) in four space-time dimensions. Considering a fundamental theory of quantum gravity and following the main concept of GR, namely that gravity is indeed geometry, it follows that there is no background metric, there is only a manifold, while geometry and matter should both have a quantum mechanical origin. This indeed differentiates LQG from other approaches of quantising gravity, which have been developed in the framework of particle physics[2]. While the full theory is not yet complete, LQG already has a number of successes, such as the construction of quantum geometry, the prediction of Planck discreteness in geometric operators, as well as a quantum accounting for black hole entropy on the horizon.

The application of ideas and mathematical methods of the full LQG theory to the cosmological sector — the dynamical variables are reduced to first homogeneity and then possibly also to isotropic models — led to Loop Quantum Cosmology (LQC) [3], which is not a field theory[3]. LQC, which gains a constantly increasing interest from the scientific community and has recently made significant progress, differs from other attempts of quantising gravity in the sense that it

[1] Background independence means that quatisation is achieved in the absence of a metric other that the physical one determined by the densitised triad. At the classical level, the fact that the laws of physics are background independent is mathematically expressed by the Einstein equations being four-diffeomorphism covariant.

[2] The only known consistent perturbative approach to quantum gravity is string theory, a theory aiming at unifying all interactions.

[3] The quantised Hamiltonian is given in terms of symmetry reduced variables, so there is only a finite number of degrees of freedom.

gets its input from the full theory of LQG. In short, it is a cosmological mini-superspace model quantised with methods of the full LQG theory. Considering a flat isotropic model within LQC, the extrinsic curvature scale $k = \dot{a} = \sqrt{8\pi G a^2 \Lambda/3}$ (a stands for the scale factor and Λ denotes a positive cosmological constant) appears in the holonomies in such a way that only $e^{i\alpha k}$, with $\alpha \in \mathbf{R}$, can be represented as operators, k itself cannot [4]. The parameter α is related to the edge length used in holonomies, which are playing the rôle of the basic operators and they imply that the Hamiltonian constraint — a Hamiltonian density which is constrained to vanish by the equations of motion — is quantised to a second order difference equation, instead of the second order differential equation of the Wheeler-De Witt (WDW) approach to quantum cosmology.

Following the approach of LQC is useful in two ways. Firstly, it allows us to get some useful insight about open issues of the full LQG theory, and secondly, by using symmetry reduction of the infinite dimensional phase space of LQG, the theory becomes often tractable leading to satisfactory answers about various interesting physical questions. The discreteness of spatial geometry, a key element of the full theory, leads to successes in LQC which do not hold on the WDW approach. In particular, it has been shown that classical big bang [5] and black hole singularities [6] are removed in LQC, in a well defined manner.

In LQC, the evolution of the universe is divided into three distinct phases depending on the value of the scales probed by the universe itself. They are the following:

- *The discrete quantum phase*: Very close to the Planck scale, the concept of space-time has no meaning and one should consider the full quantum gravity theory. Applying LQC during this phase, leads to a finite bounded spectrum for eigenvalues of inverse powers of the three-volume density, called the geometrical density.
- *The intermediate phase*: As the volume of the universe increases with time, the universe enters a semi-classical phase. More precisely, for lengths above $L_{\rm Pl} \equiv \sqrt{\gamma} l_{\rm Pl}$ ($\gamma \approx 0.2375$ is the Barbero-Immirzi parameter — a constant ambiguity parameter, whose value is fixed by black hole entropy calculations — and $l_{\rm Pl}$ is the Planck length with $l_{\rm Pl}^2 = (8\pi G)^{-1}$) the space-time can be approximated by a continuous manifold and the equations of motion take a continuous form. This intermediate phase is characterised by a second scale $L_\star$ with $L_\star \equiv \sqrt{(\gamma J \mu_0)/3} l_{\rm Pl}$, below which the geometrical density is significantly different from its classical form[4]. For scales below $L_\star$ quantum corrections cannot be neglected. This phase is the most relevant one regarding phenomenological consequences of LQC.
- *The classical phase*: At later times, and therefore large scales, the universe enters the full classical phase and standard cosmology becomes valid.

LQG/LQC is formulated in terms of SU(2) holonomies of the connection and triads[5] [7]. To obtain a quantum constraint we introduce an operator representing the curvature of the gravitational connection. A feature of LQC, which is a direct consequence of LQG, is that while there are well-defined analogs of holonomies, there is no operator corresponding to the connection; one defines a curvature operator in terms of holonomies. In the classical level, curvature can be expressed as a limit of the holonomies around a loop as the area enclosed by the loop shrinks to zero. However, in quantum geometry one cannot continuously shrink a loop to zero area, since the eigenvalues of the area operator are discrete, implying that there is a smallest non-zero area eigenvalue, called the *area gap* [8, 9]. In a canonical quantisation scheme, as the one followed in this approach, one first writes the action of GR into a Hamiltonian formulation and

[4] The half-integer J labels the ambiguity in choosing the representation in which the matter part of the Hamiltonian constraint for a scalar field is quantised. The length parameter μ_0 is related to the underlying discrete structure.

[5] The connection in LQG determines the parallel transport of chiral fermions, mathematically represented by spinors, while the conjugate momenta can be interpreted as spatial triads (i.e., *square-roots* of the metric of the 3-dimensional space).

then one quantises this classical Hamiltonian. In the *old* quantisation, the quantised holonomies were taken to be shift operators with a fixed magnitude. However, it was shown [10, 11] that this approach leads to unavoidable instabilities in the continuum semi-classical limit, where the WDW wave-function becomes a good approximation to the difference equation of LQC. For a large semi-classical universe, the WDW wave-function would be oscillating on scales of the order $(a\sqrt{\Lambda})^{-1}$. As the universe expands, this scale becomes eventually smaller than the discreteness scale of the difference equation of LQC, implying that discreteness of spatial geometry would become apparent in the behaviour of wave-functions describing a classical universe. In the underlying LQG theory, the contributions to the discrete Hamiltonian operator depend on the state which describes the universe. As the universe expands, the full Hamiltonian constraint operator creates new vertices of a lattice state, in addition to changing their edge labels. As the extrinsic curvature scale k increases with increasing volume, the corresponding α decreases since the lattice is being continuously refined. Thus, in the context of LQC one has a refinement of the discrete lattice[6]. One can choose such a lattice refinement model, so that the increase in extrinsic curvature scale k can be balanced by the decrease of α such that αk remains small and semi-classical behaviour is achieved for any macroscopic volume even in the presence of a positive cosmological constant.

Lattice refinement is also required from phenomenological reasons [11, 12]. For example, as we will discuss, lattice refinement renders a successful inflationary era[7] more natural [12]. The effect of lattice refinement has been modelled and the elimination of the instabilities in the continuum era has been explicitly shown.

The *correct* lattice refinement model should be obtained from the full LQG theory. In principle, one should use the full Hamiltonian constraint and find the way that its action balances the creation of new vertices as the volume increases. Instead, phenomenological arguments have been used, where the choice of the lattice refinement model is constrained by the form of the matter Hamiltonian [17]. In particular, we have shown [17] that for LQC to generically support inflation, and other matter fields, without the onset of large scale quantum gravity corrections, one should adopt a particular model of lattice refinement. This choice has been then found [18] to be the only one, for which physical quantities are independent of the choice of the elementary cell used to regulate the spatial integrations. Amazingly enough, this is exactly the choice required for the uniqueness of the factor ordering of the Wheeler-De Witt equation [19].

Lattice refinement leads to new dynamical difference equations which, in general, do not have a uniform step-size making their study quite involved, particularly for anisotropic cases, as for example for Bianchi models or black hole interiors. Numerical techniques have been developed [20, 21] to address this issue.

Our primary interest here concerns cosmological predictions of quantum gravity, and therefore we only focus on questions dealing with the early universe and the initial conditions. A quantum theory of gravity is expected, on the one hand, to cure the classical singularities of GR and, on the other hand, either to provide the conditions suitable for the onset of inflation, or to suggest an alternative scenario for alleviating the strong fine-tuning of the standard cosmological model.

[6] The parameter, denoted earlier by μ_0, which appears in the regularisation of the Hamiltonian constraint will no longer be constant within the lattice refinement context.

[7] It was hoped that LQC would help to overcome the extreme fine tuning necessary to achieve successful inflation in GR [13, 14, 15]. However, it has been shown [16] that semi-classical corrections are insufficient to alleviate this difficulty. Certainly, inflation could be generic in the deep quantum regime.

2. Elements of LQG/LQC

LQG, as well as LQC, are both based on a Hamiltonian formulation of GR with basic variables an SU(2) valued connection and the conjugate momentum variable which is a densitised triad[8], a derivative operator quantised in the full LQG theory in the form of fluxes. By using connection-triad variables, arising from a canonical transformation of Arnowitt-Desner-Misner (ADM) variables, we make an analogy with gauge theories; this will be helpful when dealing with quantisation issues.

The densitised triad carries information about the spatial geometry, encoded in the three-metric; the connection carries information about the spatial curvature, in the form of the spin-connection, and the extrinsic curvature. More precisely, the densitised triad, E^a_i, is related to the three-metric, q_{ab}, by

$$E^a_i = \sqrt{|q|} e^a_i \ , \tag{1}$$

where e^a_i is a physical triad, dual ($e^a_i e^j_a = \delta^j_i$) to the co-triad, e^j_a, and satisfying

$$q_{ab} = e^i_a e^j_b \delta_{ij} \ . \tag{2}$$

Note that i refers to the Lie algebra index and a is a spatial index with $a, i = 1, 2, 3$.

The connection, A^i_a, can be related to the ADM variables as

$$A^i_a = \Gamma^i_a + \gamma K^i_a \ , \tag{3}$$

where $\Gamma^i_a = -(1/2)\epsilon^{ijk} e^b_j (2\partial_{[a} e^k_{b]} + e^c_k e^l_a \partial_c e^l_b)$ stands for the spin-connection compatible with the co-triad, γ is the Barbero-Immirzi parameter — classically it has no physical consequence, while at the quantum level it plays a rôle in the level spacing of discrete geometric eigenvalues — and K^i_a stands for the extrinsic curvature one-form $K^i_a = e^{bi} K_{ab}$ (with K_{ab} the extrinsic curvature), it is the Lie derivative of q with respect to the normal vector to the spatial slice, $K^i_a = (\mathcal{L}_{\vec{n}} q_{ab}) \delta^{ij} e^b_j$.

The Poisson bracket of the densitised triad and the connection reads

$$\{A^i_a(x), E^b_j(y)\} = \kappa\gamma \delta^b_a \delta^i_j \delta^3(x, y) \ , \tag{4}$$

where $\kappa \equiv 8\pi G$.

The Hamiltonian for GR is given by the sum of constraints, with the scalar constraint

$$C_{\rm GR} = \frac{1}{2\kappa} \int_\Sigma {\rm d}^3 x N(x) \left[\frac{E^a_i E^b_j}{\sqrt{|{\rm det} E|}} \epsilon^{ij}_{\ \ k} F^k_{[ab]} - 2(1+\gamma^2) \frac{E^a_i E^b_j}{\sqrt{|{\rm det} E|}} K^i_{[a} K^j_{b]} \right] \ , \tag{5}$$

being the most important one. Note that $F^k_{[ab]} = 2\partial_{[a} A^k_{b]} + \epsilon^k_{\ ij} A^i_{[a} A^j_{b]}$ denotes the curvature two-form of the connection, and N stands for the lapse function.

Before proceeding, let us note that for any quantisation scheme based on a Hamiltonian framework, as for example LQC, or an action principle, as for example path integral approaches, for the homogeneous flat ($k = 0$) model, one should regularise the divergences which appear due to the homogeneity as the action and Hamiltonian are integrated over spatial hyper-surfaces. To do so, the spatial homogeneity and Hamiltonian are restricted to a fiducial cell[9], with finite volume $V_0 = \int {\rm d}^3 x \sqrt{|^0 q|}$ [4].

[8] The variables E^a_i and A^i_a were introduced by Barbero [22] as an alternative to the complex Ashtekar [23] variables. Both real and complex connections have been used for canonical gravity. The complex connection has SL(2,C) as gauge group, while the real connection has SU(2) as gauge group. Mathematical techniques can only cope with a quantum theory based on SU(2).

[9] Note that μ_0, which we will discuss later is the scale of the finite fiducial cell that spatial integration is restricted to, so as to remove the divergences that occur in non-compact topologies.

The expressions can be simplified a lot by restricting the analysis to homogeneous and isotropic geometries:

$$q = [a(t)]^2 \delta_{ij}\ {}^0\omega_a^i\ {}^0\omega_b^j \mathrm{d}x^a \mathrm{d}x^b\ , \tag{6}$$

where the one-forms satisfy $\partial_{[a}\ {}^0\omega_{b]}^i = 0$, leading to

$$q = [a(t)]^2[(1-kr^2)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2]\ . \tag{7}$$

The symmetry reduced variables E, A are given by

$$A_a^i = V_0^{-1/3} c\ {}^0\omega_a^i \quad , \quad E_i^a = pV_0^{-2/3}|\det\ {}^0\omega|\ {}^0e_i^a\ , \tag{8}$$

where the vector fields ${}^0e_i^a$ are dual to the 1-forms: ${}^0\omega_a^i\ {}^0e_j^a = \delta_j^i$.

Concentrating on isotropic cosmological models with constant spatial curvature, we will first consider symmetric connections and triads, and we will then insert them into the full action leading to a symmetry reduced action. The symmetric connections and triads can be decomposed using basis one-forms and vector fields obtained by Bianchi models. One can show that the actions lead to the correct equations of motion of GR, implying that they are equivalent to the Einstein-Hilbert action on metric variables.

The loop quantisation of the flat FLRW scalar constraint changes the curvature 2-form F, its ab component reads [24]:

$$F_{ab}^k = -2\ \lim_{Ar_\square \to 0} \mathrm{Tr}\left(\frac{h_{\square_{ij}}^{(\bar{\mu})} - 1}{\bar{\mu}^2 V_0^{2/3}}\right)\tau^k\ {}^0\omega_a^i\ {}^0\omega_b^j\ ; \tag{9}$$

$Ar_\square$ is the area of the square $\square_{ij}$ in the (i,j)-plane swept by a face of the elementary cell, the holonomy $h_{\square_{ij}}^{(\mu_0)}$ around the square $\square_{ij}$ is the product of holonomies along the four edges of $\square_{ij}$, and $\bar{\mu}$ is $\bar{\mu} = \sqrt{\Delta}/|p|^{1/2}$, with Δ the eigenvalue of the area operator.

In the case of a spatially flat background, derived from the Bianchi I model, the isotropic connection can be expressed in terms of the dynamical component of the connection $\tilde{c}(t)$ as

$$A_a^i = \tilde{c}(t)\omega_a^i\ , \tag{10}$$

with ω_a^i a basis of left-invariant one-forms $\omega_a^i = \mathrm{d}x^i$. The densitised triad can be decomposed using the Bianchi I basis vector fields $X_i^a = \delta_i^a$ as

$$E_i^a = \sqrt{{}^0q}\,\tilde{p}(t)X_i^a\ , \tag{11}$$

where ${}^0q$ stands for the determinant of the fiducial background metric,

$${}^0q_{ab} = \omega_a^i\omega_{bi}\ , \tag{12}$$

and $\tilde{p}(t)$ denotes the remaining dynamical quantity after symmetry reduction.

In terms of the metric variables with three-metric $q_{ab} = a^2\omega_a^i\omega_{bi}$, the dynamical quantity is just the scale factor $a(t)$. Given that the Bianchi I basis vectors are $X_i^a = \delta_i^a$,

$$|\tilde{p}| = a^2\ , \tag{13}$$

where the absolute value is taken because the triad has an orientation. Since the basis vector fields are spatially constant in the spatially flat model, the connection component is

$$\tilde{c} = \mathrm{sgn}(\tilde{p})\gamma\frac{\dot{a}}{N}\ . \tag{14}$$

Note that in what follows, the lapse function which is a constant due to spatial homogeneity, will be set equal to 1. Thus, GR can be formulated as a gauge theory in Ashtekar variables.

The canonical variables $\tilde{c}, \tilde{p}$ are related through

$$\{\tilde{c}, \tilde{p}\} = \frac{\kappa\gamma}{3} V_0 \ , \tag{15}$$

where V_0 the volume of the elementary cell adapted to the fiducial triad.

Defining the triad component p, determining the physical volume of the fiducial cell, and the connection component c, determining the rate of change of the physical edge length of the fiducial cell, as

$$p = V_0^{2/3}\tilde{p} \qquad , \qquad c = V_0^{1/3}\tilde{c} \ , \tag{16}$$

respectively, we obtain

$$\{c, p\} = \frac{\kappa\gamma}{3} \ , \tag{17}$$

independent of the volume V_0 of the fiducial cell.

To quantise, we follow the approach used in the full LQG theory. The metric itself is a physical field which must be quantised; it cannot be considered as a fixed background. Thus, to quantise gravity we use gauge theory variables to define holonomies of the connection along a given edge

$$h_e(A) = \mathcal{P} \exp \int \mathrm{d}s \dot{\gamma}^\mu(s) A^i_\mu(\gamma(s))\tau_i \ , \tag{18}$$

where $\mathcal{P}$ indicates a path ordering of the exponential, γ^μ is a vector tangent to the edge and $\tau_i = -i\sigma_i/2$, with σ_i the Pauli spin matrices, and fluxes of a triad along an S surface

$$E(S, f) = \int_S \epsilon_{abc} E^{ci} f_i \mathrm{d}x^a \mathrm{d}x^b \ , \tag{19}$$

with f_i an SU(2) valued test function. Note that even though these variables appear rather artificial, as presented here in the context of LQC, they nevertheless arise naturally within the full LQG theory.

Thus, the basic configuration variables in LQC are holonomies of the connection

$$h_i^{(\mu_0)}(A) = \cos\left(\frac{\mu_0 c}{2}\right) \mathbf{1} + 2\sin\left(\frac{\mu_0 c}{2}\right)\tau_i \ , \tag{20}$$

along a line segment $\mu_0 \ {}^0e^a_i$ and the flux of the triad

$$F_S(E, f) \propto p \ ;$$

the basic momentum variable is the triad component p. Note that $\mathbf{1}$ is the identity 2×2 matrix and $\tau_i = -i\sigma_i/2$ is a basis in the Lie algebra SU(2) satisfying the relation

$$\tau_i\tau_j = (1/2)\epsilon_{ijk}\tau^k - (1/4)\delta_{ij} \ .$$

Let us first review the *old quantisation* procedure. We follow the same approach as in LQG. We thus take $e^{i\mu_0 c/2}$ (with μ_0 an arbitrary real number) and p, as the elementary classical variables, which have well-defined analogues [4]. Using the Dirac bra-ket notation and setting $e^{i\mu_0 c/2} = \langle c|\mu\rangle$, the action of the operator $\hat{p}$ acting on the basis states $|\mu\rangle$ is

$$\hat{p}|\mu\rangle = \frac{\kappa\gamma\hbar|\mu|}{6}|\mu\rangle \ , \tag{21}$$

where μ (a real number) stands for the eigenstates of $\hat{p}$, satisfying the orthonormality relation

$$\langle\mu_1|\mu_2\rangle = \delta_{\mu_1,\mu_2} \ . \tag{22}$$

The action of the $\widehat{\exp\left[\frac{i\mu_0}{2}c\right]}$ operator acting on basis states $|\mu\rangle$ is

$$\widehat{\exp\left[\frac{i\mu_0}{2}c\right]}|\mu\rangle = \exp\left[\mu_0\frac{\mathrm{d}}{\mathrm{d}\mu}\right]|\mu\rangle = |\mu+\mu_0\rangle \ , \tag{23}$$

where μ_0 is any real number. Thus, in the old quantisation, the operator $e^{i\mu_0 c/2}$ acts as a simple shift operator.

Using the volume operator $\hat{V} = |\hat{p}|^{3/2}$, representing the volume of the elementary cell with eigenvalues $V_\mu = (\kappa\gamma\hbar|\mu|/6)^{3/2}$, we get[10] [24]

$$\hat{V}|\mu\rangle = \left(\frac{\kappa\gamma\hbar|\mu|}{6}\right)^{3/2}|\mu\rangle \ . \tag{24}$$

To define the inverse volume operator one has to trace over SU(2) valued holonomies. Since there is a freedom in choosing the irreducible representation to perform the trace, an ambiguity J arises[11]. Let us use the $J = 1/2$ irreducible representation of SU(2). The inverse volume operator is diagonal in the $|\mu\rangle$ basis and is given by [4]

$$\widehat{V^{-1}}|\mu\rangle = \left|\frac{6}{\kappa\gamma\hbar\mu_0}\left(\{V(\mu+\mu_0)\}^{1/3} - \{V(\mu-\mu_0)\}^{1/3}\right)\right|^3|\mu\rangle \ , \tag{25}$$

where μ_0 is proportional to the length of the holonomy. Note that the regulating length μ_0 is the crucial parameter in the quantum corrections; μ_0 determines the step-size of the resulting difference equation. In the above equation, Eq. (25), the eigenvalues are bounded and approach zero near the classical singularity; in the classical case the eigenvalues diverge at the singularity $\mu = 0$. The eigenvalues reach their maximum at a characteristic scale μ_0, at larger scales they approach the classical values and at smaller scales they are suppressed [25].

As in the full LQG theory, there is no operator corresponding to the connection. Nevertheless, the action of its holonomy is well-defined. Let us denote by $\hat{h}_i^{(\mu_0)}$ the holonomy along the edge parallel to the i^{th} basis vector of length $\mu_0 V_0^{1/3}$ with respect to the fiducial metric. Its action on the basis states is given by [26]

$$\hat{h}_i^{(\mu_0)}|\mu\rangle = (\widehat{\mathrm{cs}}\mathbf{1} + 2\widehat{\mathrm{sn}}\tau_{\mathrm{i}})\,|\mu\rangle \ , \tag{26}$$

where,

$$\begin{aligned} \widehat{\mathrm{cs}}|\mu\rangle \equiv \widehat{\cos(\mu_0 c/2)}|\mu\rangle &= [\ |\mu+\mu_0\rangle + |\mu-\mu_0\rangle\ ]/2 \ , \\ \widehat{\mathrm{sn}}|\mu\rangle \equiv \widehat{\sin(\mu_0 c/2)}|\mu\rangle &= [\ |\mu+\mu_0\rangle - |\mu-\mu_0\rangle\ ]/(2i) \ . \end{aligned} \tag{27}$$

Thus,

$$\begin{aligned} &\hat{h}_i^{(\mu_0)}\hat{h}_j^{(\mu_0)}\hat{h}_i^{(\mu_0)-1}\hat{h}_j^{(\mu_0)-1}|\mu\rangle \\ &= \left[\left(\widehat{\mathrm{cs}}^4 - \widehat{\mathrm{sn}}^4\right)\mathbf{1} + 2\left(\mathbf{1} - 4\tau_j\tau_i\right)\widehat{\mathrm{cs}}^2\widehat{\mathrm{sn}}^2 + 4\left(\tau_i - \tau_i\right)\mathbf{1}\widehat{\mathrm{cs}}\ \widehat{\mathrm{sn}}^3\right]|\mu\rangle \ , \end{aligned} \tag{28}$$

[10] Being concerned with the large scale behaviour of the LQC equations, we neglect the sign ambiguity.

[11] Note that $J/2$ stands for the spin of the representation. Usually one quantises the gravitational part of the Hamiltonian constraint using the fundamental $J = 1/2$ representation, and the ambiguity is only investigated for the matter part.

and

$$\hat{h}_i^{(\mu_0)}\left[\hat{h}_i^{(\mu_0)-1},\hat{V}\right]|\mu\rangle$$
$$=\left(\hat{V}-\widehat{\rm cs}\hat{V}\widehat{\rm cs}-\widehat{\rm sn}\hat{V}\widehat{\rm sn}\right)\mathbf{1}|\mu\rangle+2\tau_i\left(\widehat{\rm cs}\hat{V}\widehat{\rm sn}-\widehat{\rm sn}\hat{V}\widehat{\rm cs}\right)|\mu\rangle\ . \tag{29}$$

The gravitational part of the Hamiltonian operator in terms of SU(2) holonomies and the triad component, in the irreducible $J=1/2$ representation[12] , reads [25, 26]

$$\hat{\mathcal{C}}_{\rm grav}=\frac{2i}{\kappa^2\hbar\gamma^3\mu_0^3}{\rm tr}\sum_{ijk}\epsilon^{ijk}\left(\hat{h}_i^{(\mu_0)}\hat{h}_j^{(\mu_0)}\hat{h}i^{(\mu_0)-1}\hat{h}_j^{(\mu_0)-1}\hat{h}_k^{(\mu_0)}\left[\hat{h}_k^{(\mu_0)-1},\hat{V}\right]\right)\ . \tag{30}$$

As in the full LQG theory, curvature is defined in LQC in terms of holonomies around closed loops. This implies that the limit $\mu_0\to 0$ of the above operator does not exist, since it would mean that the area enclosed by loops should be shrunk to zero. In the underlying quantum geometry, the eigenvalues of the area operator are discrete, implying that there is a smallest nonzero eigenvalue, an *area gap* Δ [9]. This is indeed the reason for which the WDW differential equation gets replaced by a difference equation whose step size is controlled by Δ. Since μ_0 enters through the holonomies, its value in the fixed lattice case was fixed by demanding that the eigenvalue of the area operator be the area gap: $\Delta=2\sqrt{3}\pi\gamma l_{\rm Pl}^2$, implying $\mu_0=3\sqrt{3}/2$.

The action of the self-adjoint Hamiltonian constraint operator, $\hat{\mathcal{H}}_{\rm grav}=(\hat{\mathcal{C}}_{\rm grav}+\hat{\mathcal{C}}_{\rm grav}^\dagger)/2$, on the basis states, $|\mu\rangle$, is

$$\hat{\mathcal{H}}_{\rm grav}|\mu\rangle=\frac{3}{4\kappa^2\gamma^3\hbar\mu_0^3}\Big\{[R(\mu)+R(\mu+4\mu_0)]|\mu+4\mu_0\rangle-4R(\mu)|\mu\rangle+[R(\mu)+R(\mu-4\mu_0)]|\mu-4\mu_0\rangle\Big\}\ , \tag{31}$$

where

$$R(\mu)=(\kappa\gamma\hbar/6)^{3/2}\left||\mu+\mu_0|^{3/2}-|\mu-\mu_0|^{3/2}\right|\ . \tag{32}$$

Having the Hamiltonian operator, dynamics are determined by the Hamiltonian constraint[13]

$$\left(\hat{\mathcal{H}}_{\rm grav}+\hat{\mathcal{H}}_\phi\right)|\Psi\rangle=0\ . \tag{33}$$

Note that in the full LQG theory, there is an infinite number of constraints, whereas in the reduced homogeneous and isotropic case there is only one integrated Hamiltonian constraint. Matter is then introduced by just adding the actions of matter components to the gravitational action. One finally obtains difference equations analogous to the differential WDW equations.[14]

Let us impose the constraint equation on the physical wave-functions $|\Psi\rangle$, which can be expanded using the basis states as $|\Psi\rangle=\sum_\mu\Psi_\mu(\phi)|\mu\rangle$, with summation over values of μ and where the dependence of the coefficients on ϕ represents the matter degrees of freedom. Since the states $|\mu\rangle$ are eigenstates of the triad operator, the coefficients $\Psi_\mu(\phi)$ represent the state in the triad representation. Thus, quantising the Friedmann equation along the lines of the constraint in the full LQG theory, one gets the following difference equation [27]

$$\left[\left|V_{\mu+5\mu_0}-V_{\mu+3\mu_0}\right|+\left|V_{\mu+\mu_0}-V_{\mu-\mu_0}\right|\right]\Psi_{\mu+4\mu_0}(\phi)-4\left|V_{\mu+\mu_0}V_{\mu-\mu_0}\right|\Psi_\mu(\phi)$$
$$+\left[\left|V_{\mu-3\mu_0}-V_{\mu-5\mu_0}\right|+\left|V_{\mu+\mu_0}-V_{\mu-\mu_0}\right|\right]\Psi_{\mu-4\mu_0}(\phi)=-\frac{4\kappa^2\gamma^3\hbar\mu_0^3}{3}\mathcal{H}_\phi(\mu)\Psi_\mu(\phi)\ , \tag{34}$$

[12] With this choice, the Hamiltonian constraint is free of the ill-behaving spurious solutions.

[13] Note that the Gauss and the diffeomorphism constraints are automatically satisfied by appropriate gauge fixing.

[14] The reader should note that in LQC the physical fundamental object is the discrete difference equation, while the differential equation is just the approximation in the continuum limit.

where the matter Hamiltonian $\hat{\mathcal{H}}_\phi$ is assumed to act diagonally on the basis states with eigenvalue $\mathcal{H}_\phi(\mu)$. Equation (34) is indeed the quantum evolution (in internal time μ) equation. There is no continuous variable (the scale factor in classical cosmology), but a label μ with discrete steps. The wave-function $\Psi_\mu(\phi)$, depending on internal time μ and matter fields ϕ, determines the dependence of matter fields on the evolution of the universe. A massless scalar field plays the rôle of the *emergent time*. Thus, in LQC the quantum evolution is governed by a second order difference equation, rather than the second order differential equation of the WDW quantum cosmology. As the universe becomes large and enters the semi-classical regime, the WDW differential equation becomes a very good approximation to the difference equation of LQC.

3. Lattice refinement

Consider the continuum limit (namely that $\mu \gg \mu_0$) of the Hamiltonian constraint operator acting on the physical states. In the small regulating length μ_0 limit, one obtains [17] a second order difference equation which distinguishes the components of the wave-functions in different lattices of spacing $4\mu_0$. Assuming that Ψ does not vary much on scales of the order of $4\mu_0$, known as *pre-classicality* [28], one can smoothly interpolate between the points on the discrete function $\Psi_\mu(\phi)$ and approximate them by the continuous function $\Psi(\mu,\phi)$. In this way, one approximates the difference equation by a differential equation for a continuous wave-function.

The form of the wave-functions indicates that the period of oscillations can decrease as the scale increases, which implies that at sufficiently large scales the assumption of *pre-classicality* breaks down[15]. This would then lead to quantum gravity corrections at large scales (i.e., classical) physics. To avoid this undesired event, was one of the motivations behind lattice refinement [27, 29]. Allowing the length scale of the holonomies to vary, the form of the difference equation changes. Assuming that the lattice size is growing, the step-size of the difference equation is not constant in the original triad variables. The exact form of the difference equation depends on the lattice refinement used.

Consider the particular model

$$\mu_0 \to \tilde{\mu}(\mu) = \mu_0 \mu^{-1/2} \; ; \tag{35}$$

we will come back to the issue of the lattice refinement choice in a subsequent section.

The basic operators are given by replacing μ_0 with $\tilde{\mu}$. Upon quantisation [26]

$$\widehat{e^{i\tilde{\mu}c/2}}|\mu\rangle = e^{-i\tilde{\mu}\frac{\mathrm{d}}{\mathrm{d}\mu}}|\mu\rangle \; , \tag{36}$$

which is no longer a simple shift operator since $\tilde{\mu}$ is a function of μ. Changing the basis to

$$\nu = \mu_0 \int \frac{\mathrm{d}\mu}{\tilde{\mu}} = \frac{2}{3}\mu^{3/2} \; , \tag{37}$$

one gets

$$e^{-i\tilde{\mu}\frac{\mathrm{d}}{\mathrm{d}\mu}}|\nu\rangle = e^{-i\mu_0\frac{\mathrm{d}}{\mathrm{d}\nu}}|\nu\rangle = |\nu + \mu_0\rangle \; . \tag{38}$$

The volume operator acts on these basis states as

$$\hat{V}|\nu\rangle = \frac{3\nu}{2}\left(\frac{\kappa\gamma\hbar}{6}\right)^{3/2}|\nu\rangle \; , \tag{39}$$

[15] The constant lattice, in the *old quantisation* approach, does not take into account the expansion of the fiducial cell in a FLRW background.

and the self-adjoint Hamiltonian constraint operator acts as [25]

$$\hat{\mathcal{H}}_{\rm g}|\nu\rangle = \frac{9|\nu|}{16\mu_0^3}\left(\frac{\hbar}{6\kappa\gamma^3}\right)^{1/2} \times \left[\frac{1}{2}\Big\{U(\nu)+U(\nu+4\mu_0)\Big\}|\nu+4\mu_0\rangle - 2U(\nu)|\nu\rangle + \frac{1}{2}\Big\{U(\nu)+U(\nu-4\mu_0)\Big\}|\nu-4\mu_0\rangle\right], \quad (40)$$

where

$$U(\nu) = |\nu+\mu_0| - |\nu-\mu_0| \ . \quad (41)$$

Expanding $|\Psi\rangle = \sum_\nu \Psi_\nu(\phi)|\nu\rangle$ the Hamiltonian constraint reads [17]

$$\begin{aligned}&\frac{1}{2}\Big|\nu+4\mu_0\Big|\Big[U(\nu+4\mu_0)+U(\nu)\Big]\Psi_{\nu+4\mu_0}(\phi) + 2|\nu|U(\nu)\Psi_\nu(\nu)\\ &+\frac{1}{2}\Big|\nu-4\mu_0\Big|\Big[U(\nu-4\mu_0)+U(\nu)\Big]\Psi_{\nu-4\mu_0}(\phi)\\ &= -\frac{16\mu_0^3}{9}\left(\frac{6\kappa\gamma^3}{\hbar}\right)^{1/2}\mathcal{H}_\phi(\nu)\Psi_\nu(\phi) \ ; \end{aligned} \quad (42)$$

$\mathcal{H}_\phi$ stands for the matter part of the Hamiltonian, which for a massive scalar field is given by

$$\mathcal{H}_\phi = \kappa\left[\frac{P_\phi^2}{2a^3} + a^3V(\phi)\right] \ , \quad (43)$$

with P_ϕ the momentum and $V(\phi)$ the potential of the scalar field ϕ. Quantising the Hamiltonian constraint we obtain, in terms of $p = \kappa\gamma\hbar\mu/6$,

$$\sqrt{p}\frac{\partial^2\Psi(p,\phi)}{\partial p^2} + \frac{\partial^2}{\partial p^2}\left(\sqrt{p}\Psi(p,\phi)\right) - 3p^{-3/2}\frac{\partial^2\Psi(p,\phi)}{\partial\phi^2} + \frac{6}{\kappa\hbar^2}p^{3/2}V(\phi)\Psi(p,\phi) + \mathcal{O}(\mu_0) + \cdots = 0 \ , \quad (44)$$

which is a particular factor ordering of the WDW equation for a massive scalar field.

3.1. Implications for inflation

We will show that lattice refinement is essential in order to achieve a successful inflationary era. Let us consider a fixed and a dynamically varying lattice and solve in the continuum limit the second order difference equation which governs the quantum evolution. By doing so we constrain the potential of a scalar field (the inflaton) so that the continuum approximation is valid. A second constraint is imposed on the inflationary potential so that there is consistency with measurements of the cosmic microwave background temperature anisotropies on large angular scales. Combining the two constraints in a particular inflationary model, for either a fixed lattice or a given lattice refinement model, we deduce the conditions for natural and successful inflation within LQC in each of the two cases.

More precisely, let us separate the wave-function $\Psi(p,\phi)$ into $\Psi(p,\phi) = \Upsilon(p)\Phi(\phi)$ and approximate the dynamics of the inflaton field, ϕ, by setting $V(\phi) = V_\phi p^{\delta-3/2}$, where V_ϕ is a constant and $\delta = 3/2$ in the case of slow-roll, to get [17]

$$p^{-1/2}\frac{\mathrm{d}}{\mathrm{d}p}\left[p^{-1/2}\frac{\mathrm{d}}{\mathrm{d}p}\left(p^{3/2}\Upsilon(p)\right)\right] + \beta V_\phi p^\delta\Upsilon(p) = 0 \ , \quad (45)$$

with solutions [17]

$$\Upsilon\left(p\right) \approx p^{-(9+2\delta)/8}\sqrt{\frac{2\delta+3}{2\sqrt{\beta V_\phi}\pi}}\left[C_1\cos\left(x-\frac{3\pi}{2(2\delta+3)}-\frac{\pi}{4}\right)+C_2\sin\left(x-\frac{3\pi}{2(2\delta+3)}-\frac{\pi}{4}\right)\right] , \tag{46}$$

where

$$x = 4\sqrt{\beta V_\phi}(2\delta+3)^{-1}p^{(2\delta+3)/4} , \tag{47}$$

and $\beta = 96/(\kappa\hbar^2)$.

Without lattice refinement, the discrete nature of the underlying lattice would eventually be unable to support the oscillations and the assumption of pre-classicality will break down, implying that the discrete nature of space-time becomes significant on very large scales. For the end of inflation to be describable using classical GR, it must end before a scale, at which the assumption of pre-classicality breaks down and the semi-classical description is no longer valid, is reached. We will quantify this constraint.

The separation between two successive zeros of $\Upsilon\left(p\right)$ is

$$\Delta p = \frac{\pi}{\sqrt{\beta V_\phi}}p^{(1-2\delta)/4} . \tag{48}$$

For the continuum limit to be valid, the wave-function must vary slowly on scales of the order of $\mu_c = 4\tilde{\mu}$. Thus, we impose the constraint [17]

$$\Delta p > 4\mu_0\left(\frac{\kappa\gamma\hbar}{6}\right)^{3/2}p^{-1/2} , \tag{49}$$

which implies the following constraint on V_ϕ [17]

$$V_\phi < \frac{27\pi^2}{192\mu_0^2\gamma^3\kappa^2\hbar}p^{(3-2\delta)/2} . \tag{50}$$

For slow-roll inflation, $V(\phi)$ must be approximately constant, implying $\delta\approx 3/2$. For $\mu_0 = 3\sqrt{3}/2$ and $\gamma\approx 0.24$, the constraint on the inflationary potential in units of $\hbar = 1$ reads

$$V(\phi) \lesssim 2.35\times 10^{-2} l_{\rm Pl}^{-4} . \tag{51}$$

This is a softer constraint than the one imposed for fixed lattices, namely [17]

$$V_\phi \ll 10^{-28} l_{\rm pl}^{-4} , \tag{52}$$

assuming that half of inflation takes place during the classical era.

Selecting a successful and simple inflationary model, for example $V(\phi) = m^2\phi^2/2$, an additional constraint can be imposed on the potential so that the fractional over-density in Fourier space at horizon crossing is consistent with the COBE-DMR measurements. Combining the two constraints we obtain [17] for the fixed and varying lattices

$$m \lesssim 70(e^{-2N_{\rm cl}})\ M_{\rm Pl} \tag{53}$$

$$\text{and}\ \ m \lesssim 10\ M_{\rm Pl} , \tag{54}$$

respectively.

Thus, for any significant proportion of inflation to take place during the classical era, the constraint imposed on the inflaton mass is very strong, while it becomes natural once lattice refinement is taken into account.

3.2. Relation between lattice refinement model and matter Hamiltonian

A particular lattice refinement model can support only certain types of matter [12]. To prove it, let us parametrise the lattice refinement by a parameter A and the matter Hamiltonian by a parameter δ and solve the Hamiltonian constraint. The restrictions on the two-dimensional parameter space will become apparent once we impose some physical restrictions to the solutions of the wave-functions [12].

More precisely, assuming

$$\tilde{\mu} = \mu_0 \mu^A \ , \tag{55}$$

we obtain

$$\nu = \frac{\tilde{\mu}_0 \mu^{1-A}}{\mu_0(1-A)} \ . \tag{56}$$

The WDW constraint equation reads

$$\left(\hat{\mathcal{H}}_{\rm grav} + \hat{\mathcal{H}}_\phi\right)\Psi = 0 \ . \tag{57}$$

Being interested in the large scale limit, we can approximate [12] the matter Hamiltonian with $\hat{\mathcal{H}}_\phi = \hat{\nu}^\delta \hat{\epsilon}\,(\phi)$, leading to

$$\hat{\epsilon}\,(\phi)\,\Psi \equiv \epsilon\,(\phi)\,\Psi = -\nu^{-\delta}\hat{\mathcal{H}}_{\rm grav}\Psi \ . \tag{58}$$

A necessary condition for the wave-functions to be physical is that the finite norm of the physical wave-functions, which is defined by $\int_{\phi=\phi_0} d\nu|\nu|^\delta \overline{\Psi}_1\Psi_2$, must be independent of the choice of $\phi = \phi_0$. The solutions of the constraint are renormalisable provided they decay, on large scales, faster than $\nu^{-1/(2\delta)}$.

To solve the constraint equation, we need to specify the from of $\mathcal{H}_\phi$, which has in general two terms with different scale dependence. Being interested in the large scale limit, one will be the dominant term, allowing to write [12]

$$\beta\mathcal{H}_\phi = \epsilon_\nu(\phi)\nu^{\delta_\nu} \ , \tag{59}$$

where the function ϵ_ν is constant with respect to ν.

We then solve [12] the constraint equation and consider only those solutions which are the physical ones. The large scale behaviour of the wave-functions must be normalisable — a necessary condition for having physical wave-functions — and the wave-functions should preserve pre-classicality at large scales — a necessary condition for the validity of the continuum limit [12]. This procedure leads to constraints to the two-dimensional parameter space (A, δ), which we illustrate below [12] in Fig. 1 for A in the range $0 < A < -1/2$, imposed from full LQG theory considerations [11].

We thus conclude that the continuum limit of the Hamiltonian constraint equation is sensitive to the choice of model and only a limited range of matter components can be supported within a particular choice [12].

3.3. Numerical methods for solving the constraint equation for any lattice refinement model

The lattice refinement implies new dynamical difference equations, which are not expected to have a uniform step-size, leading to technical complications. This becomes apparent in the case of two-dimensional wave-functions, such as those necessary to study Bianchi models of black hole interiors. More precisely, the information needed to calculate the wave-function at a given lattice point is not provided by previous iterations. We prescribe below a method [21] based on Taylor expansion that can be used to perform this desired interpolations with a well-defined and predictable accuracy[16].

[16] A simple local interpolation scheme to approximate the necessary data points, allowing direct numerical evolution of two-dimensional systems has been also proposed in Ref. [20].

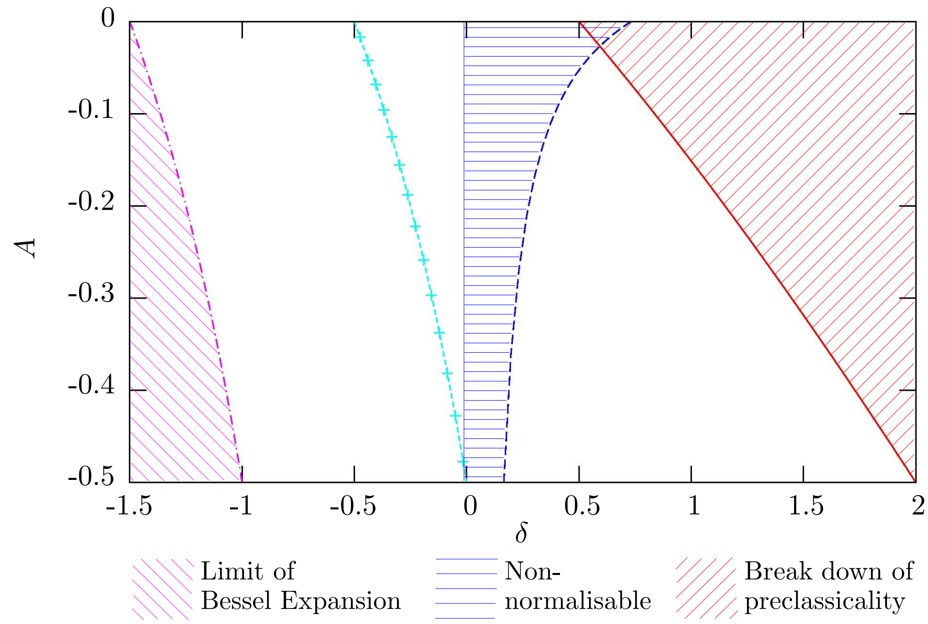

Figure 1. The allowed types of matter content are significantly restricted. Note that in the case of a varying lattice ($A \neq 0$) it is not always possible to treat the large scale behaviour of the wave-functions perturbatively (dashed line with crosses) [12].

Let us first note that the Hamiltonian constraint for a one-dimensional difference equation defined on a varying lattice, can be mapped onto a fixed lattice simply by a change of basis [21][17]. However this method is of no help for the two-dimensional case, where the Hamiltonian constraint is a difference equation on a varying lattice [11],

$$\begin{aligned}
& C_+(\mu,\tau)\left[\Psi_{\mu+2\delta_\mu,\tau+2\delta_\tau}-\Psi_{\mu-2\delta_\mu,\tau+2\delta_\tau}\right] \\
& +C_0(\mu,\tau)\left[(\mu+2\delta_\mu)\Psi_{\mu+4\delta_\mu,\tau}-2\left(1+2\gamma^2\delta_\mu^2\right)\mu\Psi_{\mu,\tau}+(\mu-2\delta_\mu)\Psi_{\mu-4\delta_\mu,\tau}\right] \\
& +C_-(\mu,\tau)\left[\Psi_{\mu-2\delta_\mu,\tau-2\delta_\tau}-\Psi_{\mu+2\delta_\mu,\tau-2\delta_\tau}\right]=\frac{\delta_\tau\delta_\mu^2}{\delta^3}\mathcal{H}_\phi\Psi_{\mu,\tau}\ ,
\end{aligned} \tag{60}$$

with

$$C_\pm \equiv 2\delta_\mu\left(\sqrt{|\tau\pm 2\delta_\tau|}+\sqrt{|\tau|}\right)\ , \tag{61}$$

$$C_0 \equiv \sqrt{|\tau+\delta_\tau|}-\sqrt{|\tau-\delta_\tau|}\ , \tag{62}$$

where we have defined δ_μ and δ_τ as the step-sizes along the μ and τ directions, respectively. The parameter δ, with $0<\delta<1$, gives the fraction of a lattice edge that the underlying graph changing Hamiltonian uses [11].

In the case of lattice refining, δ_μ and δ_τ are decreasing functions of μ and τ, respectively, and the data needed to calculate the value of the wave-function at a particular lattice site are not given by previous iterations. One can use Taylor expansions to calculate the necessary data

[17] We have checked [21] the validity of our Taylor expansion numerical method in calculating the wave-function by comparing our results with those obtained by mapping the one-dimensional difference equation defined on a varying lattice, onto a fixed lattice by performing a change of basis.

points [21]. More precisely, let us assume that the matter Hamiltonian acts diagonally on the basis states of the wave-function, namely

$$\hat{\mathcal{H}}_\phi|\Psi\rangle \equiv \hat{\mathcal{H}}_\phi \sum_{\mu,\tau} \Psi_{\mu,\tau}|\mu,\tau\rangle = \sum_{\mu,\tau} \mathcal{H}_\phi \Psi_{\mu,\tau}|\mu,\tau\rangle \ . \tag{63}$$

Given a function evaluated at three (noncolinear) coordinates, the Taylor approximation to the value at a fourth position is

$$\begin{aligned} f(x_4,y_4) &= f(x_2,y_2) + \delta^x_{42}\frac{\partial f}{\partial x}\Big|_{x_2,y_2} + \delta^y_{42}\frac{\partial f}{\partial y}\Big|_{x_2,y_2} \\ &\quad + \mathcal{O}\left((\delta^x_{42})^2\frac{\partial^2 f}{\partial x^2}\Big|_{x_2,y_2}\right) + \mathcal{O}\left((\delta^y_{42})^2\frac{\partial^2 f}{\partial y^2}\Big|_{x_2,y_2}\right) \ , \end{aligned} \tag{64}$$

where the Taylor expansion is taken about the position (x_2, y_2), we have defined $\delta^x_{ij} \equiv x_i - x_j$ and $\delta^y_{ij} \equiv y_i - y_j$, and the differentials can be approximated using

$$f(x_1,y_1) = f(x_2,y_2) + \delta^x_{12}\frac{\partial f}{\partial x}\Big|_{x_2,y_2} + \delta^y_{12}\frac{\partial f}{\partial y}\Big|_{x_2,y_2} + \cdots \ , \tag{65}$$

$$f(x_3,y_3) = f(x_2,y_2) + \delta^x_{32}\frac{\partial f}{\partial x}\Big|_{x_2,y_2} + \delta^y_{32}\frac{\partial f}{\partial y}\Big|_{x_2,y_2} + \cdots \ , \tag{66}$$

where the dots indicate higher order terms.

For slowly varying wave-functions, it has been shown [21] that linear approximation is very accurate and higher order terms in Taylor expansion can only improve the accuracy by $10^{-2}\%$. This method can be applied in any lattice refinement model, while its accuracy can be estimated. Even though we have illustrated it in the case of black hole interiors, this method can be applied to anisotropic Bianchi models and in general to systems with anisotropic symmetries.

By using this Taylor expansions method, we were able to confirm [21] numerically the stability criterion of the Schwarzchild interior, which was earlier found [11] using a von Neumann analysis, and investigate [21] how lattice refinement can change the stability properties of the system. Finally, the underlying discreteness of space-time leads to a twist [21] in the wave-functions, for both a constant lattice, as well as lattice refinement models.

3.4. Uniqueness of WDW factor ordering and the lattice refinement choice

The *correct* lattice refinement model should in principle be given by the full LQG theory. In this sense, one should consider the full Hamiltonian constraint and find the way that its action balances the creation of new vertices while the volume increases. Instead of doing so, as we have already discussed earlier, phenomenological arguments [12, 17] have been used to constrain the choice of the lattice refinement model by the form of the matter Hamiltonian. Later on, it has been argued [18] that only the lattice refinement model $\tilde{\mu} = \mu_0\mu^{-1/2}$ [19], can be achieved by physical considerations of large scale physics and consistency of the quantisation structure. We will show below that this choice is also the only one which makes the factor order ambiguities of LQC to disappear in the continuum limit [19].

Indeed, there are many ways of writing the gravitational part of the Hamiltonian constraint in terms of the triad and the holonomies of the connection, our quantisable variables. Writing [26] for example,

$$\hat{\mathcal{C}}_{\rm grav} = \frac{2i}{\kappa^2\hbar\gamma^3 k^3}\,{\rm tr}\sum_{\rm ijk}\epsilon^{\rm ijk}\left(\hat{h}_{\rm i}\hat{h}_{\rm j}\hat{h}_{\rm i}^{-1}\hat{h}_{\rm j}^{-1}\hat{h}_{\rm k}\left[\hat{h}_{\rm k}^{-1},\hat{V}\right]\right) \ , \tag{67}$$

one immediately realises that there are many possible choices of factor ordering that could have been made at this point, since classically the actions of the holonomies commute. However, each

of these factor ordering choices leads to a different factor ordering of the WDW equation in the continuum limit.

The action of the factor ordering chosen in Eq. (67) above gives [19]

$$\epsilon_{\rm ijk}{\rm tr}\left(\hat{h}_{\rm i}\hat{h}_{\rm j}\hat{h}_{\rm i}^{-1}\hat{h}_{\rm j}^{-1}\hat{h}_{\rm k}\left[\hat{h}_{\rm k}^{-1},\hat{V}\right]\right) = -24\widehat{\rm sn}^2\widehat{\rm cs}^2\left(\widehat{\rm cs}\hat{V}\widehat{\rm sn}-\widehat{\rm sn}\hat{V}\widehat{\rm cs}\right) , \tag{68}$$

while other choices have different action, as it has been explicitly found in Ref. [19].

Defining $\hat{V}|\nu\rangle = V_\nu|\nu\rangle$, the action of the above factor ordering on a general state in the Hilbert space given by $|\Psi\rangle = \sum_\nu \psi_\nu|\nu\rangle$ reads [19]

$$\begin{aligned}\epsilon_{\rm ijk}{\rm tr}\left(\hat{h}_{\rm i}\hat{h}_{\rm j}\hat{h}_{\rm i}^{-1}\hat{h}_{\rm j}^{-1}\hat{h}_{\rm k}\left[\hat{h}_{\rm k}^{-1},\hat{V}\right]\right)|\Psi\rangle &= \frac{-3i}{4}\sum_\nu\Big[\Big(V_{\nu-3k}-V_{\nu-5k}\Big)\psi_{\nu-4k}-2\Big(V_{\nu+k}-V_{\nu-k}\Big)\psi_\nu \\ &\quad +\Big(V_{\nu+5k}-V_{\nu+3k}\Big)\psi_{\nu+4k}\Big]|\nu\rangle .\end{aligned} \tag{69}$$

Similarly, one can find [19] the action of any other factor ordering choice.

Then one can take the continuum limit of these expressions by expanding $\psi_\nu \approx \psi(\nu)$ as a Taylor expansion in small k/ν. For the factor ordering chosen for this illustration here, the large scale continuum limit of the Hamiltonian constraint reads [19]:

$$\begin{aligned}&\lim_{k/\nu\to 0}\epsilon_{\rm ijk}{\rm tr}\left(\hat{h}_{\rm i}\hat{h}_{\rm j}\hat{h}_{\rm i}^{-1}\hat{h}_{\rm j}^{-1}\hat{h}_{\rm k}\left[\hat{h}_{\rm k}^{-1},\hat{V}\right]\right)|\Psi\rangle \sim \\ &\frac{-36i}{1-A}\alpha^{3/[2(1-A)]}k^3\sum_\nu \nu^{(1+2A)/[2(1-A)]}\left[\frac{{\rm d}^2\psi}{{\rm d}\nu^2}+\frac{1+2A}{1-A}\frac{1}{\nu}\frac{{\rm d}\psi}{{\rm d}\nu}+\frac{(1+2A)(4A-1)}{(1-A)^2}\frac{1}{4\nu^2}\psi(\nu)\right]|\nu\rangle .\end{aligned} \tag{70}$$

Setting $\alpha = 3\mu_0/(2k)$ and $\mu_0 = k$, all lattice refinement models will lead to the same continuum limit for the WDW equation, only for $A = -1/2$ [19], in which case the WDW equation reads [19]

$$\lim_{k/\nu\to 0}\mathcal{C}_{\rm grav}|\Psi\rangle = \frac{72}{\kappa^2\hbar\gamma^3}\left(\frac{\kappa\gamma\hbar}{6}\right)^{3/2}\sum_\nu\frac{{\rm d}^2\psi}{{\rm d}\nu^2}|\nu\rangle . \tag{71}$$

Thus, there is only one lattice refinement model, namely $\tilde{\mu} = \mu_0\mu^{-1/2}$, with a non ambiguous continuum limit[18].

In conclusion, phenomenological and consistency requirements lead to a particular lattice refinement model, implying that LQC predicts a unique factor ordering of the WDW equation in its continuum limit. Alternatively, demanding that factor ordering ambiguities disappear in LQC at the level of WDW equation leads to a unique choice for the lattice refinement model.

4. Conclusions

LQG canonically quantises space-time via triad and holonomies of the connection. Full understanding of the theory has not yet been reached, nevertheless symmetry reduction versions akin to WDW mini-superspace model have been successfully developed.

[18] The $A = -1/2$ choice can be easily understood with the following simple argument [30]. In LQC, the basis sates are $|\mu\rangle$ and the physical area of the fiducial cell is $\mu V_0^{2/3}$. Consider, in the full LQG theory, N fluxes passing through a side of the fiducial cell and divide its surface in Δ elementary surfaces. Then $N\Delta = \mu V_0^{2/3}$, implying $N = \mu V_0^{2/3}/\Delta$. The holonomies of the connection are $e^{i\lambda c/2} = e^{i\lambda\tilde{c}V_0^{1/3}/2}$ and the fiducial area is $\lambda^2 = V_0^{2/3}/N$. Thus, $\lambda \propto \mu^{-1/2}$.

As a first approximation the quantised holonomies were taken to be shift operators with a fixed magnitude. This results in the quantised Hamiltonian constraint being a difference equation with a constant interval between points on the lattice. These models lead to serious instabilities in the continuum semi-classical limit.

In the underlying LQG theory, the contributions to the discrete Hamiltonian operator depend on the state which describes the universe. As the universe expands, the number of contributions increases, so that the Hamiltonian constraint operator is creating new vertices of a lattice state, leading to a refinement of the discrete lattice in LQC.

The lattice refinement can be modelled and the instabilities in the continuum era get eliminated. We have discussed here why lattice refinement seems to be necessary to achieve a natural inflationary era, and we have illustrated that only a limited range of matter components can be supported within a particular lattice refinement choice. We have then shown that factor ordering ambiguities in the continuum limit of the gravitational part of the Hamiltonian constraint disappear only for a particular choice of lattice refinement.

Whilst the continuum limit of the lattice refinement models can be taken, there is a complication in directly evolving two-dimensional wave-functions, such as those necessary to study Bianchi models or black hole interiors. The information needed to calculate the wave-function at a given lattice point is not provided by previous iterations. We have shown that Taylor expansions can be used to perform this interpolation with a well-defined and predictable accuracy. We have then discussed how lattice refinement can alter stability conditions of the system.

We have only focused here in a few aspects of LQC concerning lattice refinement. There is certainly a much vaster arena of themes within LQC, an area which is gaining a lot of interest from the scientific community. Its basic advantage is that it allows us to successfully address some physical features of our universe, while it gives us some valuable insight for the full LQG theory.

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